OPERATIONS RESEARCH

Chapter 7
Network Analysis

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MODULE - 3: Program Evaluation and Review Technique (PERT)

3.1 Program Evaluation and Review Technique (PERT)

PERT was developed to handle projects where the time duration for each activity is not known with certainty but is a random variable which is characterized by β-distribution. Three estimates for each activity are required to calculate the expected completion time of the project:

1. **Optimistic time** (t_o) - The shortest possible time in which an activity can be performed assuming that everything goes well.

2. **Pessimistic time** (t_p) - The longest possible time required to perform an activity under extremely bad conditions. However, such conditions do not include natural calamities like earthquake, flood, etc.

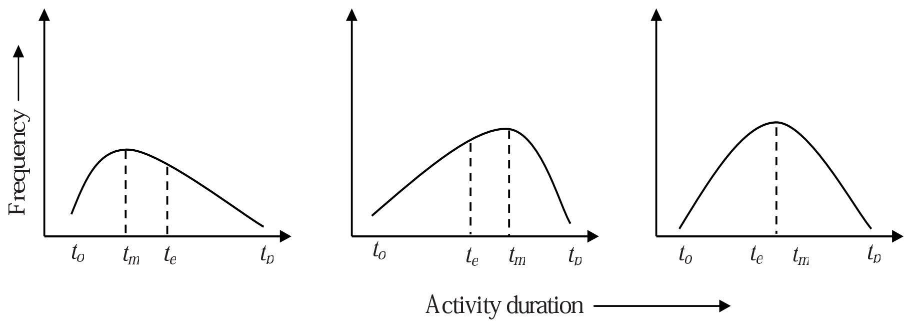

Fig. 3.1:

3. **Most likely time** (t_m) - The time that would occur most often to complete an

activity, if the activity is repeated many times under the same conditions. Obviously, it is the completion time that would occur most frequently.

4. **Expected time** (t_e) - It is the average time an activity takes if it is repeated a large number of times. The expected time is given by the formula

$$t_e = \frac{t_o + 4t_m + t_p}{6}$$

and the variance for the activity is given by the formula

$$\sigma^2 = \left(\frac{t_p - t_o}{6}\right)^2.$$

The main difference in PERT calculation is that instead of activity duration, expected time t_e for the activity is considered. With each node, variance is associated. Thus, the duration of the project is the mean expected time with variance.

3.1.1 Rules for finding variance of event

(i) Set the variance for initial event is zero, i.e., $V_1 = 0$

(ii) V_j, the variance for succeeding event j is obtained by adding activity's variance to the variance of predecessor event except at merge points, i.e., $V_j = V_i + \sigma_{ij}^2$

(iii) At merge points, the variance is computed along longest (critical) path. In the case of two paths having the same length, the larger of the two variances is chosen as the variance for that event.

Here, it is assumed that two activities are independent of each other and hence their variances can be added.

For example, Suppose that a project's expected duration is 19 days and variance of this path is 50/36. If the exact probability distribution of the path is known, it would have been easy to find out the probability of completing the project in a given time. Using Central Limit Theorem, the probability distribution of time for each event can be considered as normal. This assumption greatly simplifies calculations. Assuming normality, the probability of the project being completed by a certain date can be evaluated easily.

Prob[$Project\ duration$ 20 $days$]

$$= Prob[D \leq 20] = Prob\left[\frac{D-\mu}{\sigma} \leq \frac{20-\mu}{\sigma}\right]$$

where μ is the mean of the distribution and σ is the standard deviation.

Now, the value $(D-\mu)/\sigma$ is normalized value and usually written as z. Then

$$Prob\left[z \leq \frac{20-19}{\sqrt{50/36}}\right] = Prob[z \leq 85] = 0.80 \text{ (from standard Normal distribution table.)}$$

The probability of finishing the job in less than or equal to 20 days is 0.80. The physical meaning of this statement is as follows. If this job is done hundred times under same conditions, then there will be 80 occasions when this job will have taken 20 days or less to complete it.

One of the main advantages of PERT approach to the management of a large scale project is binding for contractual dates to finish the project. A designer would like to know the duration of the project that will have 95% chance of being completed. Let T_s be the scheduled duration such that

$$\begin{aligned} & Prob[t \leq T_s] = 0.95 \\ or, \quad & Prob\left[\frac{t-\mu}{\sigma} \leq \frac{T_s-\mu}{\sigma}\right] = 0.95 \\ or, \quad & Prob\left[Z \leq \frac{T_s-\mu}{\sigma}\right] = 0.95 \end{aligned}$$

From standard normal table, we have $z_{0.95} = 1.64$. Therefore,

$$\frac{T_s-\mu}{\sigma} = 1.64 \quad \text{gives} \quad T_s = 19 + \frac{7.07}{6} \times 1.64 = 20.90 = 21 \text{ days}$$

3.1.2 PERT Algorithm

Step 1 For a PERT network, denote the most likely time by t_m, the optimistic time t_o and pessimistic time by t_p.

Step 2 Use β-distribution for the activity duration. Compute the expected time t_e for each activity by using the formula $t_e = (t_p + 4t_m + t_o)/6$.

Step 3 Calculate the earliest start time and the latest finish time, and the total float as the difference of these two.

Step 4 Identify the critical activities and the critical path, and the expected date of completion of the project.

Step 5 Use the values of t_p and t_o and compute the variance (σ^2) of each activity's time estimate by the formula $\sigma^2 = ((t_p - t_o)/6)^2$

Step 6 Compute

$$z_0 = \frac{\text{Due date - Expected date of completion}}{\sqrt{\text{Project variance}}}$$

Step 7 Use standard normal table to find the probability $P(z \le z_0)$ of completing the project within the scheduled time, where $z = N(0,1)$.

Example 3.1: A project is represented by the network shown below and has the following data :

Task	:	A	B	C	D	E	F	G	H	I
Least time	:	5	18	26	16	15	6	7	7	3
Grestest time	:	10	22	40	20	25	12	12	9	5
Most likely time	:	8	20	33	18	20	9	10	8	4

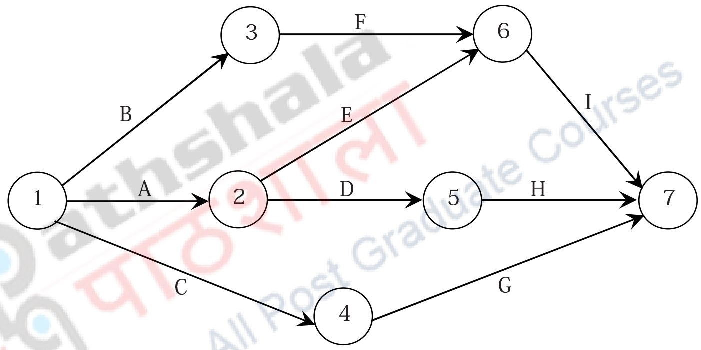

Fig. 3.2:

Determine the following :

(i) expected task time and their variance,

(ii) the earliest and latest expected times to reach each node,

(iii) the critical path, and

(iv) the probability of node occurring at the proposed completion date if the original contract time of completing the project is 41.5 weeks

Solution: (i) Using the formula $t_e = (t_o + 4t_m + t_p)/6$, $\sigma^2 = (t_p - t_o)^2/36$, we compute results given in the following table :

(ii) We find the earliest times in usual notations: $E_1 = 0$, $E_2 = 0 + 7.8 = 7.8$, $E_3 = 0 + 20 = 20$, $E_4 = 0 + 33 = 33$, $E_5 = 7.8 + 18 = 25.8$, $E_6 = \max[7.8 + 20,\ 20 + 9] = 29$, $E_7 = \max[33 + 9.8,\ 25.8 + 8,\ 29 + 4] = 42.8$.

Activity	t_o	t_p	t_m	t_e	σ^2
(1-2)	5	10	8	7.8	0.69
(1-3)	18	22	20	20.0	0.44
(1-4)	26	40	33	33.0	5.43
(2-5)	16	20	18	18.0	0.44
(2-6)	15	25	20	20.0	2.78
(3-6)	6	12	9	9.0	1.00
(4-7)	7	12	10	9.8	0.69
(5-7)	7	9	8	8.0	0.11
(6-7)	3	5	4	4.0	0.11

Table 3.1

Moving backward, we calculate the latest times as follows: $L_7 = 42.8$, $L_6 = 42.8 - 4 = 38.8$, $L_5 = 42.8 - 8 = 34.8$, $L_4 = 42.8 - 9.8 = 33$, $L_3 = 38.8 - 9 = 29.8$, $L_2 = \min[34.8 - 28,\ 38.8 - 20] = 16.8$, $L_1 = \min[16.8 - 7.8,\ 29.8 - 20,\ 33 - 33] = 0$.

(iii) To find the critical path, we calculate slack time by taking difference between the earliest expected time and the latest allowable time. Results are given in the following table and the critical path is shown by double line in figure 3.3.

Node (i)	t_e	E_i	L_i	Slack	Var. σ_i
2	7.8	7.8	16.8	9.0	0.69
3	20.0	20.0	29.8	9.8	0.44
4	33.0	33.0	33.0	0.0	5.42
5	18.0	25.8	34.8	9.0	1.13
6	9.0	29.0	38.8	9.8	1.44
7	9.8	42.8	42.8	0.0	6.12

Table 3.2

(iv) The scheduled time of completing the project is 41.5 weeks. Therefore, the distance in standard deviations that the schedule time from the earliest expected time E_i is given by

$$D_i = \frac{ST_i - E_i}{\sqrt{[Var.(i)]}} = \frac{41.5 - 42.8}{\sqrt{(6.12)}} = -0.52$$

where ST_i denotes the schedule time.

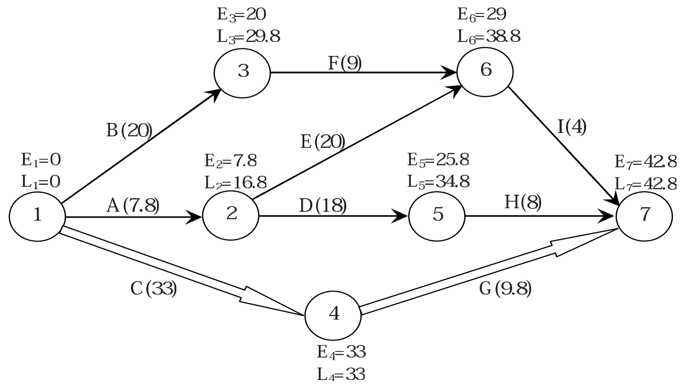

Fig. 3.3:

Therefore, $P(z \geq -0.52) = 1 - P[z \leq 0.52] = 1 - 0.70 = 0.30$ (from Normal table) which is the area under the standard normal curve bounded by ordinates at $x = 0$, and $x = 0.52$.

From this, we conclude that if the project is performed 100 times under the same conditions, there will be 30 chances when this job would take 41.5 weeks or less to complete it.

Example 3.2: A small project consists of the following activities and time estimates :

Activity	Most optimistic time	Most likely time	Most pessimistic time
1-2	4	8	12
1-3	4	10	12
1-4	8	14	24
2-5	5	8	10
3-4	2	5	8
3-5	2	4	8
4-5	6	10	14
5-6	1	3	6

Determine the following :

(i) Construct the operational network.

(ii) Locate the critical path.

(iii) Calculate the mean and standard deviation for the critical path.

(iv) What is the probability of completing the project in more than 26 weeks?

(v) Along the non-critical path 1 – 4 – 6, what is the probability of exceeding 26 weeks?

(vi) What is the probability of exceeding 26 weeks along non-critical path 1 – 2 – 5 – 6?

Solution: The operational network is as shown below.

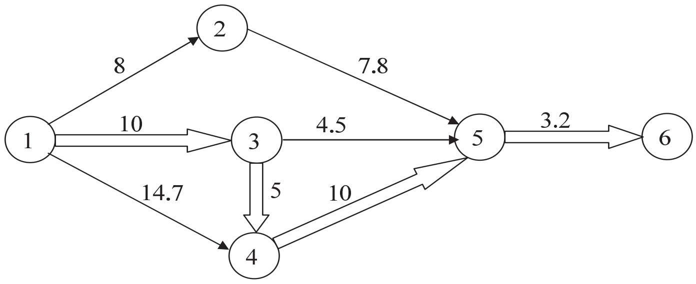

Fig. 3.4:

We now note the time estimates of the activities as shown in the following table:

Activity	t_o	t_m	t_p	$t_e = \frac{t_o+4t_m+t_p}{6}$	$\sigma = \frac{t_p-t_o}{6}$
A_{1-2}	4	8	12	8	1.3
A_{1-3}	8	10	12	10	0.6
A_{1-4}	8	14	24	14.6	2.6
A_{2-5}	5	8	10	7.8	0.8
A_{3-4}	2	5	8	5	1
A_{3-5}	2	4	8	4.3	1
A_{4-5}	6	10	14	10	1.3
A_{5-6}	1	3	6	3.1	0.8

Route	Time (weeks)	Total time (weeks)
1-2-5-6	8 +7.8 + 3.2	19
1-3-5-6	10 + 4.3 + 3.2	17.5
1-3-4-5-6	10 + 5 +10 + 3.2	28.2*
1-4-5-6	14.7 + 10 +3.2	27.9

It is evident from the above table that the critical path is 1 – 3 – 4 – 5 – 6 and $t_e = 28.2$ weeks.

Activity	t_e	σ	σ^2
A_{1-3}	10	0.7	0.47
A_{3-4}	5	1	1.00
A_{4-5}	10	1.3	1.69
A_{5-6}	3.2	0.8	0.64

Thus we have $t_e = 28.2$, $\sigma_{t_e}^2 = 3.82$ and $\sigma_{t_e} = \sqrt{3.82} = 1.954$.
Hence the mean and the standard deviation for the critical path are respectively 28.2 weeks and 1.954.

Taking $T_D = 26$, we find

$$z = \frac{26 - 28.2}{1.954} = -\frac{2.2}{1.954} = -1.12.$$

Therefore, $P(z \leq -1.12) = 13.6\%$
Hence the probability of completing the project in more than 26 weeks is $(100-13.6)\%$ $= 86.4\%$.

Along the noncritical path 1-4-5-6, we have the following results:

Activity	t_e	σ	σ^2
A_{1-4}	14.7	2.7	7.29
A_{4-5}	10	1.3	1.69
A_{5-6}	3.2	0.8	0.64

Thus we have $t_e = 27.9$, $\sigma_{t_e}^2 = 9.62$ and $\sigma_{t_e} = \sqrt{9.62} = 3.101$.
Taking $T_D = 26$, we find

$$z = \frac{26 - 27.9}{3.101} = -\frac{1.9}{3.101} = -0.6.$$

Therefore, $P(z \leq -0.6) = 27.4\%$
Therefore, the probability of exceeding 26 weeks along the non-critical path 1-4-5-6 is $(100 - 27.4)\%$ or, 72.6%.

For the noncritical path 1-2-5-6 we have the following results:

Activity	t_e	σ	σ^2
A_{1-2}	8	1.3	1.69
A_{2-5}	7.8	0.8	0.64
A_{5-6}	3.2	0.8	0.64

Thus we have $t_e = 19$, $\sigma_{t_e}^2 = 2.97$ and $\sigma_{t_e} = \sqrt{2.97} = 1.724$.
Putting $T_D = 26$, we find

$$z = \frac{26 - 19}{1.724} = +\frac{7}{1.724} = 4.06$$

From standard normal table, we find that the probability of completing the project along the path 1-2-5-6 in less than or equal to 26 weeks is cent percent. Hence the probability of exceeding 26 weeks along the noncritical path 1-2-5-6 is nil.